LOGARITHMIC JACOBIAN IDEALS, QUASI-ORDINARY HYPERSURFACES AND EQUISINGULARITY

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Abstract. We describe the jacobian ideal of the fibers S_t of an equiresolvable deformation of a quasi-ordinary hypersurface singularity $(S, 0)$. This kind of deformation, inspired by the work of Teissier, has generic fiber isomorphic to $(S, 0)$ and special fiber a toric singularity. We show a formula, in terms of the logarithmic jacobian ideal, for the pull-back of the jacobian ideal of S_t in its normalization. The logarithmic jacobian ideal is studied in the normal toric case by Lejeune and Reguera in relation with the study of motivic invariants and arc spaces. We deduce some equisingularity properties of the normalized Nash modification of S_t.

Introduction

An equidimensional germ $(S, 0)$ of an algebroid variety of dimension d is quasi-ordinary if there exists a finite map $\pi : (S, 0) \to (\mathbf{C},^d, 0)$, which is unramified outside a normal crossing divisor in $(\mathbf{C}^d, 0)$. This class of singularities plays an important role in the classical approach to study arbitrary singularities by means of the Jung's approach (see [L1] and [A]). In particular, resolution of surfaces singularities has been obtained by this method.

We suppose in this paper that the hypersurface germ $(S, 0)$ is analytically irreducible of dimension $d \geq 2$. The germ S has certain fractional power series parameterizations, possessing a finite set of *characteristic monomials*, which generalize the Newton Puiseux parameterizations of plane branches and their characteristic exponents. In the complex analytic case, Joseph Lipman and Yih-Nan Gau proved that these monomials determine many of the geometrical or topological features of S, for instance the characteristic monomials classify the embedded topological type of $(S, 0) \subset (\mathbf{C}^{d+1}, 0)$ (see [L1, Gau]).

Quasi-ordinary singularities are better understood when we study them using toric geometry methods: for instance the normalization $\nu : (Z, 0) \to (S, 0)$ is a toric singularity determined by the characteristic monomials and $(Z, 0)$ (see [GP3, PP1]). The germ $(S, 0)$ shares many features with another germ of toric variety $(Z^\Gamma, 0)$ which is a complete intersection of the same dimension, defined by a semigroup Γ determined by the characteristic monomials. We have that $Z \neq Z^\Gamma$ in general. The semigroup Γ is independent of the choice of parametrization of $(S, 0)$, see [PP1, GP2, GP-GS, GP-H].

After re-embedding the germ $(S, 0)$ in an affine space of bigger dimension, we have a deformation $(S_t, 0)$ with generic fiber (resp. special fiber) isomorphic to the germ $(S, 0)$ (resp. to $(Z^\Gamma, 0)$). The family S_t is equiresolvable with one toric modification of the ambient space, all the fibers have the same normalization $\bar{S}_t = Z$ (see [GP1] and [GP3]). The Poincaré series associated to S_t by the essential divisorial valuations is independent of t (see [GP-H]). This approach is inspired by the work of Rebeca Goldin and Bernard Teissier in the case of plane

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branches [G-T]. See also Teissier's development of this program in terms of valuations [T2]. For the different notions of equisingularity, in particular the relation between Zariski's dimensional type with quasi-ordinary singularities, see [L2].

The Nash modification of an algebraic (or complex analytic) variety is a canonical morphism which replaces each singular point by the limiting positions of tangent spaces at non singular points. In the case of complex surfaces, a finite sequence of normalized Nash modifications, i.e. Nash modifications followed by normalization, provides a resolution of singularities (see [Sp] and [H]). The Nash modification of a complete intersection singularity coincides with the blow up of the jacobian ideal (see [No] and [P]). The Nash modification of an affine toric variety is the blow up of the *logarithmic jacobian ideal* (see Section 1, [GS, LJ-R, T3]).

We define the logarithmic jacobian ideal $J_d(V)$ of a d-dimensional variety V with toric normalization $\bar{V}$ as the image of the module of differentials Ω^d_V by the composite of the maps:

$$\Omega^d_V \longrightarrow \Omega^d_{\bar{V}} \longrightarrow \Omega^d_{\bar{V}}(\log D) \longrightarrow \mathcal{O}_{\bar{V}},$$

where $\Omega^d_{\bar{V}}(\log D)$ is the module of d-forms with logarithmic poles on the complement D of the torus of $\bar{V}$, the first map is induced by the normalization, the second map is the canonical from the toric structure of $\bar{V}$ and the third is an isomorphism. This definition is motivated by the ideas of Monique Lejeune-Jalabert and Ana Reguera in the normal toric case [LJ-R].

In the influential paper [D-L] Jan Denef and François Loeser associated a *motivic Poincaré series* $P^V_{\mathrm{geom}}(T)$ to any variety V. If the variety is defined over an algebraically closed field of characteristic zero they proved that this series has a rational form by developing the theory of motivic integration over arc spaces. This subtle invariant is not very well understood. Lejeune and Reguera have shown that the logarithmic jacobian ideal of a normal affine toric surface plays a significant role in the rational expression of the associated motivic Poincaré series. In a joint work with Helena Cobo we have shown that a similar situation happens in the cases of affine toric varieties [C-GP2] and of quasi-ordinary hypersurface singularities [C-GP3], of of arbitrary dimension (see [C-GP1] for an announcement of part of the results). In particular, in these cases the logarithmic jacobian ideal determines the *motivic volume* of the arc space. Some recent results connecting arc spaces and valuations in terms of the geometry of the Nash modification appear in [dF-E-I].

Inspired by the toric case, we analyze the behavior of the family S_t under the normalized Nash modification, relative to the parameter t. Since S_t is a complete intersection, its normalized Nash modification coincides with the composite of the normalization map of S_t, followed by the normalized blow up of the pull-back of the jacobian ideal $\mathrm{Jac}(S_t)$ of S_t in Z (see [LJ-T]). Our main result is that $\mathrm{Jac}(S_t)\,\mathcal{O}_Z$ is a monomial ideal of the toric algebra $\mathcal{O}_Z$ of the form:

$$\mathrm{Jac}(S_t)\,\mathcal{O}_Z = X^{\gamma^0}\, J_d(S_t)\,\mathcal{O}_Z,$$

where the monomial X^{γ^0} is determined explicitly in terms of the characteristic monomials of $(S,0)$ (see Theorem 3.4). The term X^{γ^0} appears also in the formula $P_\Gamma(X) = (-1)^d X^{\gamma^0} P_{-\Gamma}(X)$, where $P_\Gamma(X) = \sum_{\gamma\in\Gamma} X^\gamma$ is the generating series of the semigroup Γ (see [GP-H]). This formula, which takes into account that $P_\Gamma(X)$ is a rational function, reflects the fact that the coordinate ring of Z^Γ is a Gorenstein domain (see Chapter 1, Theorem 12.7, [Sta]).

We determine the generators of the logarithmic jacobian ideals $J_d(S_t)\mathcal{O}_Z$ in terms of the characteristic monomials of S. It turns out that $J_d(S_t)\mathcal{O}_Z$ is a monomial ideal of the toric variety Z. If $t \neq 0$ the ideal $J_d(S_t)\mathcal{O}_Z$ is independent of t and is different in general of $J_d(S_0)\mathcal{O}_Z$ (see Theorem 3.3).

Our result implies that the geometry of the normalized Nash modification of a topologically equisingular family of quasi-ordinary hypersurfaces with special fiber equal to S, is the same for all the fibers of the family, i.e., it is determined in the same manner by the characteristic monomials. However, the geometry of the normalized Nash modification of the equiresolvable family considered above, is not the same in general for the generic fiber S_t, $t \neq 0$, which is isomorphic to S, and the special fiber $S_0 = Z^{\Gamma}$. We quantify this difference by giving a formula for the number $\bar{\nu}_{\mathrm{Jac}(S_t)}(\mathfrak{m}(S_t))$, where $\mathfrak{m}(S_t)$ denotes the maximal ideal of the analytic algebra $\mathcal{O}_{S_t}$. Recall that the number $\bar{\nu}_I(I')$ is defined and studied in general by Lejeune-Jalabert and Teissier in [LJ-T] for a pair $I', I \neq A$ of ideals of a commutative ring A. In the analytic case the study of this number has applications related with the properties of analytic arcs, polar varieties and Łojasiewicz exponents (see [LJ-T], [T1] and [Hi]). In Proposition 1.5 we describe the invariant $\bar{\nu}_I(I')$ for monomial ideals of a toric variety. We deduce a formula for the number $\bar{\nu}_{\mathrm{Jac}(S_t)}(\mathfrak{m}(S_t))$ in terms of the divisorial valuations associated with the irreducible components of the pull-back of the jacobian ideal of S_t in the normalized Nash modification. In Section 3.4 we give an example of family S_t such that $\bar{\nu}_{\mathrm{Jac}(S_t)}(\mathfrak{m}(S_t))$ provides different numbers for $t \neq 0$ and $t = 0$.

The results and proofs of this paper hold in the category of complex analytic spaces and in the algebroid category over an algebraically closed field of zero characteristic, which we denote by $\mathbf{C}$.

The structure of the paper is as follows: in Section 1 we review the construction of the Nash modification of a toric singularity and we give a combinatorial description of the jacobian ideal. In Section 2 we review the basic properties of quasi-ordinary hypersurface singularities and of the deformation S_t. Finally, in Section 3 we give the proofs of the main results stated above.

Notations. The result of erasing the term r_j in the sequence $r_1, \dots, r_j, \dots, r_s$ is denoted by $r_1, \dots, \hat{r}_j, \dots, r_s$. Let $A = (a^i_j)^{i=1,\dots,k}_{j=1,\dots,l}$ be a matrix with coefficients on a ring. If $1 \leq i_1 < \cdots < i_s \leq k$ (resp. if $1 \leq j_1 < \cdots < j_r \leq l$) we denote by $A^{i_1,\dots,i_s}$ (resp. by $A_{j_1,\dots,j_r}$) the submatrix:

$$A^{i_1,\dots,i_s} := (a^i_j)^{i \in \{1,\dots,\hat{i}_1\dots,\hat{i}_s,\dots,k\}}_{j\in\{1,\dots l\}} \quad (\text{resp. } A_{j_1,\dots,j_r} := (a^i_j)^{i\in\{1,\dots,k\}}_{j\in\{1,\dots,\hat{j}_1,\dots,\hat{j}_r,\dots,l\}}).$$

If A is an square matrix we denote by $|A|$ its determinant.

1. On the Nash modification of an affine toric variety

We give some definitions and notations (see [Fu], [Stu] and [T2] for proofs).

1.1. Equivariant embedding of an affine toric variety.

1.1.1. *Lattices and cones.* If $N \cong \mathbf{Z}^d$ is a lattice we denote by M the dual lattice, by $N_{\mathbf{R}}$ (resp. $N_{\mathbf{Q}}$) the vector space spanned by N over the field $\mathbf{R}$ (resp.over $\mathbf{Q}$). We use similar notations for lattice homomorphisms ϕ and the associated vector spaces homomorphisms $\phi_{\mathbf{Q}}$ or $\phi_{\mathbf{R}}$, etc. We use subscripts to denote the coordinates of a vector $v = (v_1, \dots, v_d)$ with respect to a fixed basis. We use super-scrips to label sequences $v^1, \dots v^r$ of vectors. In what follows a *cone* means a *rational convex polyhedral cone*: the set of non negative linear combinations of vectors $a^1, \dots, a^r \in N$. The cone σ is *strictly convex* if σ contains no linear subspace of dimension > 0. The *dual* cone $\sigma^{\vee}$ of σ is the set $\{w \in M_{\mathbf{R}} \mid \langle w, u\rangle \geq 0\}$. We denote by $\mathring{\sigma}$ the *relative interior* of the cone σ.

1.1.2. *Affine toric varieties.* Let Λ be a sub-semigroup of finite type of the lattice $M = \Lambda + (-\Lambda)$ which it generates as a group. The $\mathbf{C}$-algebra of the semigroup $\mathbf{C}[\Lambda] := \{\sum a_\lambda X^\lambda \mid a_\lambda \in \mathbf{C}\}$ is of finite type and corresponds to the affine toric variety of the form $Z^\Lambda = \operatorname{Spec} \mathbf{C}[\Lambda]$. The torus Z^M is an open dense subset of Z^Λ, which acts on Z^Λ in a way which extends the group multiplication on $T(N)$. We have a bijection $\tau \mapsto \mathrm{orb}_\tau$ between the faces of σ and the orbits of the torus action which reverses the inclusion of the closures. Any affine toric variety is defined by a semigroup of this form and is not necessarily normal. The cone $\sigma^\vee := \mathbf{R}_{\geq 0}\Lambda \subset M_{\mathbf{R}}$ is rational for the lattice M, hence the semigroup $\sigma^\vee \cap M$ is of finite type. The map of $\mathbf{C}$-algebras $\mathbf{C}[\Lambda] \to \mathbf{C}[\sigma^\vee \cap M]$ induced by $\Lambda \hookrightarrow \sigma^\vee \cap M$, corresponds geometrically to the normalization map $Z_\sigma := Z^{\sigma^\vee \cap M} \to Z^\Lambda$.

From now on we suppose that the cone $\sigma^\vee := \mathbf{R}_{\geq 0}\Lambda$ is strictly convex of dimension d. In this situation the zero dimensional orbit of the torus action is reduced to the point of $0 \in Z^\Lambda$ defined by the ideal $\mathfrak{m}_\Lambda := (\Lambda - \{0\})\mathbf{C}[\Lambda]$ which is maximal in this case. We denote by $\mathbf{C}[[\Lambda]]$ the ring of formal power series with coefficients in $\mathbf{C}$ and exponents in the semigroup Λ. The ring $\mathbf{C}[[\Lambda]]$ is isomorphic to the completion of the local ring of germs of holomorphic functions at $(Z^\Lambda, 0)$ with respect to its maximal ideal (see [GP3]).

1.1.3. *Normalized blow up of a monomial ideal.* By abuse of notation we denote with the same symbol a subset $\neq I \subset \sigma^\vee \cap M$ and the monomial ideal $(X^u / u \in I)$ of $\mathbf{C}[\sigma^\vee \cap M]$. The *Newton polyhedron* of the monomial ideal I is the convex hull $\mathcal{N}(I)$ of the Minkowski sum of sets $I + \sigma^\vee$. We denote by $\mathrm{ord}_{\mathcal{I}}$ the support function of the polyhedron $\mathcal{N}(I)$, which is defined by, $\mathrm{ord}_I : \sigma \to \mathbf{R}$, $\nu \mapsto \inf_{\omega \in \mathcal{N}(I)} \langle \nu, \omega \rangle$. The *dual fan* $\Sigma(\mathcal{I})$ associated to the polyhedron $\mathcal{N}(I)$ is consists of the cones $\sigma(\mathcal{F}) := \{\eta \in \sigma \mid \langle \eta, v \rangle = \mathrm{ord}_I(\eta),\ \forall v \in \mathcal{F}\}$, for $\mathcal{F}$ running through the faces of $\mathcal{N}(I)$. If $\Sigma = \Sigma(I)$, we have a toric modification $\pi_\Sigma : Z_\Sigma \to Z_\sigma$, which is the *normalized blow up* of Z_σ centered at the monomial ideal I of $\mathbf{C}[\sigma^\vee \cap M]$ (see [K-K-M-S], Chapter I, Section 2 or [LJ-R]).

1.1.4. *Equivariant embeddings.* Let $\alpha^1, \dots, \alpha^m \in \Lambda$ be a set of generators of the semigroup. The homomorphism of semigroups:

$$\phi : \mathbf{Z}^m_{\geq 0} \to \Lambda, \quad u^i \mapsto \alpha^i,$$

(where $\{u^i\}_{i=1}^m$ denotes the canonical basis of $\mathbf{Z}^m$) corresponds to an equivariant embedding $Z^\Lambda \hookrightarrow \mathbf{C}^m$ given by:

$$\iota : \mathbf{C}[U_1, \dots, U_m] \to \mathbf{C}[\Lambda], \quad U_i \mapsto X^{\alpha^i}, \quad \text{for } i = 1, \dots, m. \tag{1}$$

Let n be a vector in $\ker(\phi_{\mathbf{Z}})$, with $\phi_{\mathbf{Z}} : \mathbf{Z}^m \to M$ the lattice homomorphism induced by ϕ. The vector n is then of the form $n = n^+ - n^-$ where n^+ (resp. $-n^-$) is the vector obtained by considering only the positive (resp. negative) coordinates of $n = (n_1, \dots, n_m)$, with respect to the canonical basis of $\mathbf{Z}^m$. If $n^\pm = (n_1^\pm, \dots, n_m^\pm)$ then we have the relation:

$$k(n) := \sum_{i=1}^m n_i^+ \alpha^i = \sum_{i=1}^m n_i^- \alpha^i. \tag{2}$$

Let $n^1, \dots n^p$ be a sequence of vectors generating $\ker(\phi_{\mathbf{Z}})$. We associate to them the *matrix of relations*:

$$R := \begin{pmatrix} n_1^1 & \cdots & n_m^1 \\ \cdots & \cdots & \cdots \\ n_1^p & \cdots & n_m^p \end{pmatrix}, \tag{3}$$

which is of rank $m - d \leq p$. We have an exact sequence:

$$\mathbf{Z}^p \xrightarrow{\psi} \mathbf{Z}^m \xrightarrow{\phi_{\mathbf{Z}}} M \longrightarrow 0 \tag{4}$$

in which the first map ψ applies the i^{th}-canonical vector to the i^{th}-row vector of R, for $i = 1, \dots, p$. There exists a sequence of vectors $n^1, \dots, n^p$ generating $\ker(\phi_{\mathbf{Z}})$, such that the ideal of the embedding $Z^\Lambda \hookrightarrow \mathbf{C}^m$ is generated by $\{h_j = U^{n^{j+}} - U^{n^{j-}}\}_{j=1,\dots,p}$.

Without loss of generality we relabel the relations in such a way that the submatrix $R^{m-d+1,\dots,p}$ of R is of rank $m-d$ (cf. Notations in page 3).

Lemma 1.1. *If $R^{m-d+1,\dots,p}$ is of rank $m-d$ then for any sequence $1 \leq j_1 < \dots < j_d \leq m$ we have that:*

$$\alpha^{j_1} \wedge \dots \wedge \alpha^{j_d} = 0 \Leftrightarrow |R^{m-d+1,\dots,p}_{j_1,\dots,j_d}| = 0.$$

Proof. We consider the exact sequence induced by (4) over the field $\mathbf{Q}$. The subspace $\ell := \mathrm{Im}(\psi_{\mathbf{Q}})$, is generated by the images of the first $m-d$ canonical vectors by hypothesis. The matrix associated to the restriction of the linear map $\psi'_{\mathbf{Q}}$ to the subspace spanned by the first $m-d$ canonical vectors, with respect to the canonical basis, is equal to the transpose of $R^{m-d+1,\dots,d}$.

The matrix P associated to $\phi_{\mathbf{Q}}$ with respect to the canonical basis of $\mathbf{Q}^m$ and a fixed basis of $M_{\mathbf{Q}}$ has i^{th}-column the coordinates of $\alpha^i \in M_{\mathbf{Q}}$, for $i = 1, \dots, m$. By dualizing the sequence (4) we notice that the j^{th}-row of the matrix P defines the coordinates of an element $w^j \in (\mathbf{Q}^m)^*$, with respect to the dual basis, in such a way that the vector subspace ℓ is also obtained by the intersection of the kernels of w^j, for $j = 1, \dots, d$.

We have exhibited the $m-d$-dimensional subspace $\ell := \mathrm{Im}(\psi'_{\mathbf{Q}})$, by equations or by generators. We deduce the assertion as a consequence of the classical relations of the Grassmann coordinates of a linear subspace and its dual Grassmann coordinates applied to ℓ (see [H-P], VII, §3, Theorem 1). □

1.2. **On the Nash modification of a toric singularity.** In the case of an affine toric variety the Nash modification is isomorphic to the blow up of a monomial ideal, called the logarithmic jacobian ideal in [LJ-R], described in terms of a sequence of generators of the semigroup. In the normal case this description is due to Gonzalez-Sprinberg [GS]. It generalizes also to the non necessarily normal case, as shown in a recent non published work of Teissier [T3], which we give in this Section by convenience to the reader. Lejeune-Jalabert and Reguera have given recently another proof of this result in the normal toric case, in terms of the logarithmic jacobian ideal, by using differential forms with equivariant logarithmic poles (see [LJ-R]). We consider a semigroup Λ as in Section 1.1.

Notation 1.2. *We denote by J (resp. $\tilde{J}$) the matrix with coefficients in $\mathbf{C}[\Lambda]$ defined from the jacobian matrix of $(h_1, \dots, h_p)$ by:*

$$J := \left(\iota(\frac{\partial h_i}{\partial U_j}) \right)^{i=1,\dots,p}_{j=1,\dots,m}, \quad (resp. \tilde{J} := \left(\iota(U_j \frac{\partial h_i}{\partial U_j}) \right)^{i=1,\dots,p}_{j=1,\dots,m}),$$

where the map ι is defined by

Remark 1.3. *By (2) and logarithmic differentiation we have that the matrix $\tilde{J}$ is equal to the result of multiplying the i^{th}-row of R by the monomial $X^{k(n^i)}$, for $i = 1, \dots p$.*

Proposition 1.4. *(see* [GS], [T3] *and* [LJ-R], *Appendix) The Nash modification of Z^Λ is the blow up of the ideal of $\mathbf{C}[\Lambda]$ generated by the images of the products $U_{j_1} \dots U_{j_d}$ such that $\alpha^{j_1} \wedge \dots \wedge \alpha^{j_d} \neq 0$, i.e., by the monomial ideal*

$$\{\alpha^{j_1} + \dots + \alpha^{j_d} \mid 1 \leq j_1 < \dots < j_d \leq m \text{ and } \alpha^{j_1} \wedge \dots \wedge \alpha^{j_d} \neq 0\}. \tag{5}$$

Proof. Nobile shows that the Nash modification of a complete intersection X is the blow up of the image of the jacobian ideal of X in its coordinate ring $\mathcal{O}_X$ (see[No], Theorem 1, Remark 2).

We suppose first that the toric singularity Z^Λ is a complete intersection. This means that $p = m - d$ and the map ψ in the sequence (4) is injective. By Remark 1.3 we have the following Formula, for $1 \leq j_1 < \cdots < j_d \leq m$:

$$X^{\alpha^1 + \cdots + \alpha^m} \, |J_{j_1,\ldots,j_d}| = X^{k(n^1)+\cdots+k(n^{m-d})} X^{\alpha^{j_1}+\cdots+\alpha^{j_d}} \, |R_{j_1,\ldots,j_d}|. \tag{6}$$

Formula (6) implies that the *jacobian ideal of* Z^Λ, generated by $\{|J_{j_1,\ldots,j_d}|\}_{1\leq j_1 < \cdots < j_d \leq m}$, and the ideal generated by $\{X^{\alpha^{j_1}+\cdots+\alpha^{j_d}} \, |R_{j_1,\ldots,j_d}|\}_{1\leq j_1<\cdots<j_d\leq m}$ are related by invertible ideals hence they have isomorphic blow ups. Finally, by Lemma 1.1 we have that the number $|R_{j_1,\ldots,j_d}|$ vanishes if and only if $\alpha^{j_1} \wedge \cdots \wedge \alpha^{j_d} = 0$, for $1 \leq j_1 < \cdots < j_d \leq m$.

In the general case, by the proof of Theorem 1 of [No] the Nash modification of an irreducible variety X of dimension d is isomorphic to the blow up of the restriction to X of the jacobian ideal of a d-dimensional suitable complete intersection X' containing X. In our situation $X = Z^\Lambda$ and we can take by X' the binomial variety defined by $h_1 = 0, \ldots, h_{m-d} = 0$, corresponding to a submatrix of relations $R^{m-d+1,\ldots,m}$ of rank $m - d$. Notice that if we replace in (6) the minor $R_{j_1,\ldots,j_d}$ by $R^{m-d+1,\ldots,m}_{j_1,\ldots,j_d}$ and $J_{j_1,\ldots,j_d}$ by $J^{m-d+1,\ldots,m}_{j_1,\ldots,j_d}$ then the resulting formula holds. The proof follows from the same argument by using Lemma 1.1. ☐

Proposition 1.5. *If* $1 \leq j_1 < \cdots < j_d \leq m$ *and* $1 \leq i_1 < \cdots < i_{m-d} \leq p$ *we set*

$$m^{i_1,\ldots,i_{m-d}}_{j_1,\ldots,j_d} := k(n^{i_1}) + \cdots + k(n^{i_{m-d}}) + \sum_{j=1,\ldots,\hat{j}_1,\ldots,\hat{j}_d,\ldots,m} \alpha^j.$$

The jacobian ideal Jac(Z^Λ) *of the toric variety* Z^Λ *is the monomial ideal of* $\mathbf{C}[\Lambda]$ *defined by the set* $\{m^{i_1,\ldots,i_{m-d}}_{j_1,\ldots,j_d}\}^{1\leq i_1<\cdots i_{m-d}\leq p}_{1\leq j_1<\cdots<j_d\leq m} \subset \Lambda$, *where the sequence* $1 \leq j_1 < \cdots < j_d \leq m$ *(resp.* $1 \leq i_1 < \cdots < i_{m-d} \leq p$*) verifies that* $\alpha^{j_1} \wedge \cdots \wedge \alpha^{j_d} \neq 0$ *(resp. the rank of the matrix* $R^{1,\ldots,\hat{i}_1,\ldots,\hat{i}_{m-d},\ldots,p}$ *is equal to* $m - d$*).*

Proof. Notice that the jacobian ideal of Z^Λ is generated by $\{|J^{1,\ldots,\hat{i}_1,\ldots,\hat{i}_{m-d},\ldots,p}_{j_1,\ldots,j_d}|\}^{1\leq i_1<\cdots i_{m-d}\leq p}_{1\leq j_1<\cdots<j_d\leq m}$. By Formula (6) in the proof of Proposition 1.4, we have that this generator is equal to 0 if $|(R_0)^{1,\ldots,\hat{i}_1,\ldots,\hat{i}_{m-d},\ldots,p}_{j_1,\ldots,j_d}| = 0$, otherwise it is equal to a non zero constant times the monomial with exponent $m^{i_1,\ldots,i_{m-d}}_{j_1,\ldots,j_d}$, where the rank of the matrix $R^{1,\ldots,\hat{i}_1,\ldots,\hat{i}_{m-d},\ldots,p}$ is equal to $m - d$ and $\alpha^{j_1} \wedge \cdots \wedge \alpha^{j_d} \neq 0$ by Lemma 1.1. ☐

1.3. **An analytic invariant.** We recall the definition of the number $\bar{\nu}_I(I')$ associated to a pair of ideals $I, I' \neq A$ of a ring A by Lejeune-Jalabert and Teissier (see [LJ-T]). This number has many interesting applications in the frame of analytic geometry, in particular in connections with the Łojasiewicz exponents and analytic arcs (see [LJ-T], [T1] and [Hi]) . We show how to describe this number explicitly when A is the coordinate ring of an affine toric variety Z^Λ and I and I' are a pair of monomial ideals of $A = \mathbf{C}[\Lambda]$. In particular, if $I = \mathrm{Jac}(Z^\Lambda)$ and $I' = \mathfrak{m}_\Lambda$ is the maximal ideal defining the distinguished point of Z^Λ, the number $\bar{\nu}_{\mathrm{Jac}(Z^\Lambda)}(\mathfrak{m}(\Lambda)$ is an analytic invariant of the germ of the singularity $(Z^\Lambda, 0)$.

Let A be a ring and $I \neq A$ be an ideal. Associated to the I-adic filtration $A = I^0 \supset I = I^1 \supset I^2 \supset \cdots$ we consider the order function ν_I defined by $\nu_I(a) := \sup\{k \mid a \in I^k\} \in \mathbf{Z}_{\geq 0} \cup \{\infty\}$, for $a \in A$. If I' is an ideal of A we set $\nu_I(I') := \sup\{k / I' \subset I^k\}$. We set $\bar{\nu}_I(a) := \sup_{k\in\mathbf{Z}_{>0}} \frac{\nu_I(a^k)}{k}$ and $\bar{\nu}_I(I') := \sup_{k\in\mathbf{Z}_{>0}} \frac{\nu_I(I')}{k}$. If $I' = (a_1, \ldots, a_n)$ we have that $\bar{\nu}_I(I') := \min_{i=1,\ldots,n} \bar{\nu}_I(a_i)$. These numbers belong to $\mathbf{R}_{\geq 0} \cup \{\infty\}$ (see [LJ-T]).

Suppose from now on that $A = \mathbf{C}[[\Lambda]]$ and I, I' are monomial ideals of $\mathbf{C}[[\Lambda]]$, for Λ as in Section 1.1. By Proposition 0.2.10 of [LJ-T] we have that $\bar{\nu}_I(I') = \bar{\nu}_{I\mathbf{C}[[\sigma^\vee\cap M]]}(I'\mathbf{C}[[\sigma^\vee \cap M]])$, hence we can suppose that $\Lambda = \sigma^\vee \cap M$ is a saturated semigroup.

Let $\Sigma = \Sigma(I)$ be the dual fan associated to the Newton polyhedron of I. Then, the normalized blow up of I in Z_σ is the toric modification $Z_\Sigma \to Z_\sigma$. Let $\{D_k\}_{k\in K}$ be the irreducible components of the divisor defined by $I\mathcal{O}_{Z_\Sigma}$. Each component D_k is invariant for the torus action and the divisorial valuation ν_{D_k} defined by D_k is a monomial valuation of the form $\nu_{D_k}(\sum a_m X^m) := \inf_{a_m\neq 0}\langle n_k, m\rangle$, where n_k is a primitive vector for the lattice N in an edge ρ of Σ such that, if τ is the unique face of σ such that $\rho \subset \overset{\circ}{\tau}$, the orbit orb_τ is contained in the zero locus of I in Z^Λ. See [LJ-R, K-K-M-S, GP-GS].

Proposition 1.6. *With the previous notations we have that:*

$$\bar{\nu}_I(I') = \min_{k\in K} \frac{\mathrm{ord}_{I'}(n_k)}{\mathrm{ord}_I(n_k)}.$$

Proof. By the results of §4.1 in [LJ-T] we have that if $\phi \in \mathbf{C}[[\sigma^\vee \cap M]]$ then $\bar{\nu}_I(\phi) = \min_{k\in K} \frac{\nu_{D_k}(\phi)}{\nu_{D_k}(I\mathcal{O}_{Z_\Sigma})}$. It follows from the combinatorial description of the normalized blow up of I that $\nu_{D_k}(I\mathcal{O}_{Z_\Sigma}) = \mathrm{ord}_I(n_k)$. If $I' = (X^{m_1}, \dots, X^{m_r})$ we have that $\nu_{D_k}(X^{m_i}) = \langle \nu_k, m_i\rangle$ hence we deduce that $\bar{\nu}_I(I') = \min_{i=i,\dots,m_r} \min_{k\in K}\langle n_k, m_i\rangle (\mathrm{ord}_I(n_k))^{-1} = \min_{k\in K} \mathrm{ord}_{I'}(n_k)(\mathrm{ord}_I(n_k))^{-1}$. □

2. Quasi-ordinary hypersurface singularities

An algebroid hypersurface germ $(S, 0) \subset (\mathbf{C}^{d+1}, 0)$ is *quasi-ordinary* if there exists a finite projection $(S, 0) \to (\mathbf{C}^d, 0)$ which is a local isomorphism outside a normal crossing divisor. In the hypersurface case the embedding $(S, 0) \subset (\mathbf{C}^{d+1}, 0)$ can be defined by an equation $f = 0$ where $f \in \mathbf{C}[[X]][X_{d+1}]$ is a *quasi-ordinary polynomial*: a Weierstrass polynomial with discriminant $\Delta_Y f$ of the form $\Delta_Y f = X^\delta \epsilon$ for a unit ϵ in the ring $\mathbf{C}[[X]]$ of formal power series in the variables $X = (X_1, \dots, X_d)$ and $\delta \in \mathbf{Z}^d_{\geq 0}$.

We suppose in this paper that the germ $(S, 0)$ is analytically irreducible. The Jung-Abhyankar theorem guarantees that the roots of a quasi-ordinary polynomial f, called *quasi-ordinary branches*, are fractional power series in the ring $\mathbf{C}[[X^{1/n_0}]]$, for $n_0 = \deg f$ (see [A]). If the series $\{\zeta^{(l)}\}_{l=1}^{\deg f} \subset \mathbf{C}\{X^{1/n_0}\}$ are the roots of f, the discriminant $\Delta_Y f$ of f with respect to Y is equal to $\Delta_Y f = \prod_{i\neq j}(\zeta^{(i)} - \zeta^{(j)})$, hence each factor $\zeta^{(t)} - \zeta^{(r)}$ is of the form $X^{\lambda^{t,r}}\epsilon_{t,r}$ where $\epsilon_{t,r}$ is a unit in $\mathbf{C}[[X^{1/n_0}]]$. The monomials $X^{\lambda^{t,r}}$ (resp. the exponents $\lambda^{t,r}$) are called *characteristic* The characteristic exponents can be relabelled in the form

$$\lambda^1 < \lambda^2 < \cdots < \lambda^g, \tag{7}$$

where $<$ means $\leq$ coordinate-wise and $\neq$ (see [L1]). The characteristic exponents determine the following nested sequence of lattices: $M_0 \subset M_1 \subset \cdots \subset M_g =: M$ where $M_0 := \mathbf{Z}^d$ and $M_j := M_{j-1} + \mathbf{Z}\lambda^j$ for $j = 1, \dots, g$ with the convention $\lambda^{g+1} = +\infty$. The index n_j of the lattice M_{j-1} in the lattice M_j is > 1 (see [L1, GP2]).

2.1. **Normalization of a quasi-ordinary singularity.**

We study quasi-ordinary singularities by using toric geometry methods.

The semigroup $\mathbf{Z}^d_{\geq 0}$ has a minimal set of generators $e^1, \dots, e^d$, which is a basis of the lattice M_0. The dual basis of the dual lattice N_0 spans a regular cone σ in $N_{0,\mathbf{R}}$. Then, we have that $\sigma^\vee = \mathbf{R}_{\geq 0}e^1 + \cdots + \mathbf{R}_{\geq 0}e^d$ and $\mathbf{Z}^d_{\geq 0} = \sigma^\vee \cap M_0$. The $\mathbf{C}$-algebra of formal power series $\mathbf{C}[[X_1, \dots, X_d]]$ is isomorphic to $\mathbf{C}[[\sigma^\vee \cap M_0]]$. This isomorphism transforms the monomial $X_1^{\alpha_1} \cdots X_d^{\alpha_d}$ in the *monomial* $X^\alpha \in \mathbf{C}[\sigma^\vee \cap M_0]$, where $\alpha = \sum_{k=1}^d \alpha_k e^k$.

We identify the local algebra $\mathcal{O}_S = \mathbf{C}[[X_1, \dots, X_d]][X_{d+1}]/(f)$ of the singularity $(S, 0)$ with the ring $\mathbf{C}[[X_1, \dots, X_d]][\zeta] = \mathbf{C}[[\sigma^\vee \cap M_0]][\zeta]$. The normalization of a quasi-ordinary singularity (non necessarily hypersurface) is analytically isomorphic to a toric simplicial singularity (see [PP2]). In the hypersurface case the normalization is determined by the characteristic exponents as follows.

Lemma 2.1. (see Proposition 14 of [GP3]) *We have that the quasi-ordinary branch ζ belongs to $\mathbf{C}[[\sigma^\vee \cap M]]$. The homomorphism of $\mathbf{C}$-algebras $\mathcal{O}_S = \mathbf{C}[[\sigma^\vee \cap M_0]][\zeta] \to \mathbf{C}[[\sigma^\vee \cap M]]$ is the inclusion of $\mathcal{O}_S$ in its integral closure $\mathcal{O}_{\bar{S}} = \mathbf{C}[[\sigma^\vee \cap M]]$ in its field of fractions.* ☐

2.2. **The associated toric variety Z^Γ.** Set

$$\bar\lambda^1 = \lambda^1 \quad \text{and} \quad \bar\lambda^{j+1} = n_j \bar\lambda^j + \lambda^{j+1} - \lambda^j, \quad \text{for} \quad j = 1, \dots, g-1. \tag{8}$$

The semigroup

$$\Gamma := \mathbf{Z}_{\geq 0} e^1 + \cdots + \mathbf{Z}_{\geq 0} e^d + \bar\lambda^1 \mathbf{Z}_{\geq 0} + \cdots + \bar\lambda^g \mathbf{Z}_{\geq 0} \subset \sigma^\vee \cap M$$

is associated to a sequence of characteristic exponents $\lambda^1, \dots, \lambda^g$ of a quasi-ordinary hypersurface, following the analogy to the case of plane branches (see [GP2] and [K-M]). The semigroup Γ is an analytical invariant of the singularity $(S, 0)$ (see [GP2], [PP1], [GP-GS] and [GP-H]). In the analytic case the semigroup Γ is a complete invariant of the embedded topological type of $(S, 0) \subset (\mathbf{C}^{d+1}, 0)$, as characterized by the work of Lipman and Gau (see [L1] and [Gau]).

Notation 2.2. *We denote by $e^1, \dots, e^{d+g}$ (resp. by $\bar e^1, \dots, \bar e^{d+g}$) the sequence of vectors $e^1, \dots, e^d$, $\lambda^1, \dots, \lambda^g$ (resp. $e^1, \dots, e^d, \bar\lambda^1, \dots, \bar\lambda^g$). By convenience we denote $\bar\lambda_{g+1} = \infty$.*

Notation 2.3. *In what follows we consider the coordinates $(\gamma_1, \dots, \gamma_d)$ of a vector $\gamma \in M$ with respect to the canonical basis $\mathcal{B} := \{\bar e^1, \dots, \bar e^d\}$ of $M_{\mathbf{Q}}$.*

We consider the partial order $\leq$ on M given by

$$\gamma \leq \gamma' \Leftrightarrow \gamma_k \leq \gamma'_k, \text{ for } k = 1, \dots, d.$$

The notation $\gamma < \gamma'$ means that $\gamma \leq \gamma'$ and $\gamma \neq \gamma'$. By (7) and (8) we have that $\bar\lambda^1 < \cdots < \bar\lambda^g$ and by Lemma 3.3 of [GP2] we have that:

$$n_j \bar\lambda^j \in \textstyle\sum_{r=1}^{d+j-1} \mathbf{Z}_{\geq 0} \bar e_r \subset M_{j-1} \quad \text{and} \quad n_j \bar\lambda^j \; < \; \bar\lambda^{j+1}, \text{ for } j = 2, \dots, g. \tag{9}$$

Lemma 2.4. (See Lemma 3.3 of [GP2]) *The generators of the semigroup Γ verify relations of the form:*

$$-l^{(i)}_{d+i} \bar e^{d+i} = l^{(i)}_1 \bar e^1 + \cdots + l^{(i)}_d \bar e^d + l^{(i)}_{d+1} \bar e^{d+1} + \cdots + l^{(i)}_{d+i-1} \bar e^{d+i-1}, \tag{10}$$

where $0 \leq l^{(i)}_r$ for $r = 1, \dots d$, $0 \leq l^{(i)}_{d+k} < n_k$ for $k = 1, \dots, i-1$ and $l^{(i)}_{d+i} = -n_i$, for $i = 1, \dots, g$.

We consider the equivariant embedding of the toric variety $Z^\Gamma \subset \mathbf{C}^{d+g}$ given by $U_i = X^{\bar e^i}$ for $i = 1, \dots, d+g$, where $(U_1, \dots, U_{d+g})$ denotes coordinates at the origin of $\mathbf{C}^{d+g}$.

Proposition 2.5. (see Proposition 38, [GP3]) *The binomials*

$$h_i \; := \; U_1^{l_1^{(i)}} \cdots U_{d+i-1}^{l^{(i)}_{d+i-1}} \; - \; U_{d+i}^{n_i}, \text{ for } i = 1, \dots, g, \tag{11}$$

define the ideal of $Z^\Gamma \subset \mathbf{C}^{d+g}$. The toric variety Z^Γ is a complete intersection.

The matrix of relations associated with (11) is:

$$R := \begin{pmatrix} l_1^{(1)} & \cdots & l_d^{(1)} & l_{d+1}^{(1)} & 0 & 0 & \cdots & 0 \\ l_1^{(2)} & \cdots & l_d^{(2)} & l_{d+1}^{(2)} & l_{d+2}^{(2)} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ l_1^{(g)} & \cdots & l_d^{(g)} & l_{d+1}^{(g)} & l_{d+2}^{(g)} & l_{d+3}^{(g)} & \cdots & l_{d+g}^{(g)} \end{pmatrix} \tag{12}$$

We study some elementary properties related with this matrix:

Definition 2.6. *For $1 \leq k \leq d$ we denote by m_k the smallest integer $1 \leq m \leq g+1$ such that $l_k^{(m)} \neq 0$, where by convention $m_k = g+1$ means that $l_k^{(1)} = \cdots = l_k^{(g)} = 0$.*

Lemma 2.7. *If $1 \leq k \leq d$, the k^{th}-coordinate of a vector $w = \sum_{r=1}^{d+m_k-1} a_r \bar{e}^r \in M_{m_k-1}$ with respect to the basis $\mathcal{B}$ is a_k, in particular if $1 \leq i < m_k$ then $\bar{\lambda}_k^i = 0$ and if $w \in \sigma^\vee \cap M_{m_k-1}$ then $a_k \in \mathbf{Z}_{\geq 0}$.*

Proof. Notice that the lattice M_{m_k-1} splits as the direct sum of the following two lattices $\mathbf{Z}e^k$ and $\sum_{r=1,\ldots,d+m_k-1}^{r\neq k} \mathbf{Z}\bar{e}_r \subset \oplus_{r=1,\ldots,d}^{r\neq k} \mathbf{Q}e^r$. □

Lemma 2.8. *We have that $\lambda^j \leq \bar{\lambda}^j$ and $\lambda_k^j \neq 0$ implies that $n_j \bar{\lambda}_k^j \geq 1$.*

Proof. The inequality $\lambda_k^j \leq \bar{\lambda}_k^j$ follows trivially by definition and an easy induction. For the second inequality, if $j=1$ it holds since $n_1\bar{\lambda}^1 = n_1\lambda^1 \in M_0$ thus the coordinate $n_1\bar{\lambda}_k^1$ is a positive integer whenever $\lambda_k^1 \neq 0$. Suppose that the result is true for $j-1$. If $\lambda_k^{j-1} \neq 0$ then by the induction hypothesis we have that $1 \leq n_{j-1}\bar{\lambda}_k^{j-1}$ and $n_{j-1}\bar{\lambda}_k^{j-1} < \bar{\lambda}_k^j$ by (9). Otherwise $\lambda_k^{j-1} = 0$ and this implies that $\lambda_k^r = 0$, for $r = 1, \ldots, j-1$ by (7). It follows that $m_k = j$ and $\bar{\lambda}_k^j = \lambda_k^j > 0$ by (10). Since $n_j\bar{\lambda}^j \in M_{j-1}$ by (9), we deduce from Lemma 2.7 that $n_j\bar{\lambda}_k^j$ is an integer ≥ 1. □

2.3. **Equiresolvable deformation.** In [GP1] we construct an embedding $(S,0) \hookrightarrow (\mathbf{C}^{d+g},0)$ together with a deformation $(\mathcal{S}_t,0) \subset (\mathbf{C}^{d+g},0)$ of $(S,0)$, with generic fiber S_t for $t \neq 0$, isomorphic to $(S,0)$ and special fiber $S_0 = (Z^\Gamma,0)$. This deformation is equisingular in the sense that there exists a toric modification $\pi : W \to \mathbf{C}^{d+g}$, characterized by the properties of the semigroup Γ, such that it provides simultaneous normalization of $(S_t,0)$ (see [GP3]). Then, the composite of π with any toric resolution of the singularities of W is an embedded resolution of singularities of $(S_t,0)$, for all $t \in \mathbf{C}$. This result generalizes one of Teissier and Goldin for plane branches (see [G-T]).

The equisingular deformation S_t has equations of the following form:

Proposition 2.9. *(see [GP1], Proposition 3.15 and 3.17 and [GP3] Proposition 39). The monomial U_i is given the weight $\bar{e}^i \in M$, for $i = 1, \ldots, d+g$. The ideal of $(S_t,0) \subset (\mathbf{C}^{d+g},0)$ is generated by elements of the form:*

$$H_i = h_i + c_i U_{d+i} + \sum_{\underline{r}=(r_1,\ldots,r_{d+i-1},0\ldots,0)} c_{\underline{r}}^{(i)} U_1^{r_1} \cdots U_{d+i-1}^{r_{d+i-1}}, \text{ for } i = 1, \ldots, g, \tag{13}$$

where $U_{d+g+1} := 0$, the coefficients $c_i, c_{\underline{r}}^{(i)}$ are monomials in $(t)\mathbf{C}[t]$ and the terms $U_1^{r_1} \ldots U_{d+i-1}^{r_{d+i-1}}$ have weight $> n_i\bar{\lambda}_i$, for $i = 1, \ldots, g$.

We denote by

$$\bar{\iota} : \mathbf{C}[[U_1, \ldots, U_{d+g}]] \to \mathbf{C}[[\sigma^\vee \cap M]], \tag{14}$$

the composition of the canonical maps $\mathbf{C}[[U_1, \ldots, U_{d+g}]] \to \mathcal{O}_{S_t}$ with the inclusion of $\mathcal{O}_{S_t}$ in its integral closure $\mathcal{O}_{\bar{S}_t} = \mathbf{C}[[\sigma^\vee \cap M]]$. Notice that this holds for all $t \in \mathbf{C}$ (see [GP3] for the case $t = 0$).

3. The image of the jacobian ideal in the integral closure and the logarithmic jacobian ideal

We consider the equiresolvable deformation $\mathcal{S}_t$ with fibers S_t introduced in Section 2.3. We study the image of the jacobian ideal of S_t in the analytic algebra $\mathcal{O}_{\bar{S}_t}$, for $t \in \mathbf{C}$ a parameter.

We compare this ideal with the logarithmic jacobian ideal $J_d(S_t)$, which is also an ideal of $\mathbf{C}[[\sigma^\vee \cap M]]$.

3.1. **The definition of the logarithmic jacobian ideal.** Let Y be the algebroid germ defined by an ideal I of $\mathbf{C}[[X_1, \dots, X_n]]$ with analytic algebra $A = \mathbf{C}[[X_1, \dots, X_n]]/I$. We denote by Ω^1_Y the A-module of *Kähler differential forms* and by $d : A \to \Omega^1_Y$ its canonical derivation. As usual we denote by Ω^k_Y the A-module $\Omega^k_Y := \bigwedge^k \Omega^1_Y$. See [G-L-S] Chapter I, §1.10.

We introduce the logarithmic jacobian ideal of a singularity with toric normalization following [LJ-R] in the toric case. We review the normal toric case following [Od] Chapter 3 and [LJ-R] Appendix. We consider the toric singularity Z with analytic algebra $\mathcal{O}_Z = \mathbf{C}[[\sigma^\vee \cap M]]$. We denote by D the equivariant Weil divisor defined by the sum of orbit closures of codimension one in the toric variety Z. The $\mathcal{O}_Z$-module $\Omega^1_Z(\log D)$ of 1-forms of Z with logarithmic poles along D is identified with $\mathcal{O}_Z \otimes_{\mathbf{Z}} M$. We have a map of $\mathcal{O}_Z$-modules $\eta : \Omega^1_Z \to \mathcal{O}_Z \otimes_{\mathbf{Z}} M$, determined by $dX^\gamma \mapsto X^\gamma \otimes \gamma$, for $\gamma \in \sigma^\vee \cap M$. Notice that if $\{\gamma_i\}_{i=1}^r$ generate the semigroup $\sigma^\vee \cap M$ then $\{dX^{\gamma_i}\}_{i=1}^r$ generate the $\mathcal{O}_Z$-module Ω^1_Z. If $\phi = \sum_{\gamma \in \sigma^\vee \cap M} c_\gamma X^g$ then $d\phi \mapsto \omega(d\phi) = \sum_{\gamma \in \sigma^\vee \cap M} c_\gamma X^g \otimes \gamma$. Notice that $\omega(d\phi) = \sum_{i=1}^d (\sum_{\gamma \in \sigma^\vee \cap M} c_\gamma \, \gamma_i X^g) \otimes u_i$, where $(\gamma_1, \dots \gamma_d)$ denote the coordinates of γ in terms of a basis $u_1, \dots, u_d$ of the lattice M. For $k = 1, \dots, d$ we have that $\wedge^k \eta$ is an homomorphism of $\mathcal{O}_Z$-modules:

$$\wedge^k \eta : \Omega^k_Z \longrightarrow \Omega^k_Z(\log D) = \mathcal{O}_Z \otimes_{\mathbf{Z}} \bigwedge^k M, \quad dX^{\gamma_1} \wedge \cdots \wedge dX^{\gamma_k} \mapsto X^{\gamma_1 + \cdots + \gamma_k} \otimes \gamma_1 \wedge \cdots \wedge \gamma_k.$$

Notice that fixing a basis $u_1, \dots, u_d$ of the rank d lattice M provides an isomorphism $\phi : \bigwedge^d M \to \mathbf{Z}$, given by $u_1 \wedge \cdots \wedge u_d \mapsto 1$. This provides an identification of $\Omega^d_Z(\log D)$ with $\mathcal{O}_Z$. It follows that $\eta^d(\Omega^d_Z)$ corresponds by this identification to an ideal of $\mathcal{O}_Z$ independent of the basis of M chosen. This ideal is called the *logarithmic jacobian ideal* of Z in [LJ-R]. It is a monomial ideal generated by $(X^{\gamma_{j_1} + \cdots + \gamma_{j_d}})^{1 \leq j_1 < \cdots < j_d \leq r}_{\gamma_{j_1} \wedge \cdots \wedge \gamma_{j_d} \neq 0}$.

Definition 3.1. *Let W be an analytically irreducible algebroid germ of singularity of dimension d such that its normalization Z is a toric singularity. The* logarithmic jacobian ideal $J_d(W)$ of *W is the $\mathcal{O}_W$-module generated by the image of the module of differentials Ω^d_W by the composite τ of the maps:*

$$\Omega^d_W \longrightarrow \Omega^d_Z \xrightarrow{\wedge^d \eta} \Omega^d_Z(\log D) \xrightarrow{\phi} \mathcal{O}_Z, \tag{15}$$

where $\Omega^d_W \to \Omega^d_Z$ is the map induced by the normalization.

Remark 3.2. *If Z is the normalization of the affine toric variety Z^Λ, for Λ a semigroup as in Section 1.1, then $J_d(Z^\Lambda)$ is the ideal of $\mathbf{C}[\Lambda]$ generated by (5).*

3.2. **Computation of the logarithmic jacobian ideal of S_t.** We consider the equiresolvable family S_t introduced in Section 2.3. By Lemma 2.1 for $t \neq 0$ and [GP3] for $t = 0$ we have an inclusion $i_t : \mathcal{O}_{S_t} \to \mathcal{O}_Z = \mathbf{C}[[\sigma^\vee \cap M]]$ corresponding to the normalization map $Z = Z^{\sigma^\vee \cap M} \to S_t$. Notice that the normalization Z is a toric singularity independent of the parameter t. We study the logarithmic jacobian ideal $J_d(S_t)$ of S_t.

Theorem 3.3. *We use Notation 2.2. The ideal $J_d(S_t)\mathcal{O}_Z$ is a monomial ideal of $\mathcal{O}_Z = \mathbf{C}[[\sigma^\vee \cap M]]$ defined by the subsets of $\sigma^\vee \cap M$:*

$$\{\bar{e}^{j_1} + \cdots + \bar{e}^{j_d} \mid \bar{e}^{j_1} \wedge \cdots \wedge \bar{e}^{j_d} \neq 0\} \quad \textit{if } t = 0, \textit{ and} \tag{16}$$

$$\{e^{j_1} + \cdots + e^{j_d} \mid e^{j_1} \wedge \cdots \wedge e^{j_d} \neq 0\}^{d < j_d \leq d+g}_{1 \leq j_1 < \cdots < j_{d-1} \leq d} \quad \textit{if } t \neq 0. \tag{17}$$

Proof of Theorem 3.3. If $t = 0$ the result is a consequence of Remark 3.2 and Propositions 1.4 and 1.5. We prove the result for $t \neq 0$. By definition of the deformation S_t in Section 2.3 we can embed $(S_t, 0) \subset (\mathbf{C}^{d+1}, 0)$ with coordinates $(X_1, \dots, X_{d+1})$ by eliminating recursively $U_{d+g}, \dots U_{d+2}$ in the equations (13) using that $t \neq 0$ and setting $X_i := U_i$, for $i = 1, \dots, d$. By [GP3] the restriction to S_t of the projection $(X_1, \dots X_{d+1}) \to (X_1, \dots, X_d)$ is a quasi-ordinary projection, and the characteristic monomials $\lambda_1, \dots, \lambda_g$ of a quasi-ordinary branch ζ_t parameterizing S_t are independent of t for all $t \neq 0$.

We have the inclusion $i_t : \mathcal{O}_{S_t} \to \mathcal{O}_{\bar{S}_t}$ corresponding to the normalization map, where $\mathcal{O}_{\bar{S}_t} = \mathbf{C}[[\sigma^\vee \cap M]]$ by Lemma 2.1. We denote by x_i the image of the coordinate X_i in $\mathcal{O}_{S_t}$, for $i = 1, \dots d+1$. By Lemma 2.1, $i_t(x_{d+1}) = \zeta_t$ belongs to $\mathbf{C}[[\sigma^\vee \cap M]]$ and we have $i_t(x_i) = X^{\bar{e}^i}$ for $i = 1, \dots, d$. We consider the expansion $\zeta_t = \sum_{\lambda \in \sigma^\vee \cap M} c_\lambda X^\lambda$ in $\mathbf{C}[[\sigma^\vee \cap M]]$.

If $1 \leq k \leq d$ we consider the index m_k of Definition 2.6. We decompose the series ζ_t as $\zeta_t = \zeta_{t,k}^- + \zeta_{t,k}^+$ where $\zeta_{t,k}^- = \sum_{\lambda_{m_k} \not\leq \lambda} c_\lambda X^\lambda$ and $\zeta_{t,k}^+ = \sum_{\lambda_{m_k} \leq \lambda} c_\lambda X^\lambda$. By Lemma 2.7 we have that if λ appears in the series $\zeta_{t,k}^-$ then the k-coordinate of λ with respect to the basis $\mathcal{B}$ of $M_\mathbf{Q}$ is an integer ≥ 0 (see Notation 2.3).

For a sequence $1 \leq j_1 < \dots < j_d \leq d+1$ we analyze the image of $dx_{j_1} \wedge \dots \wedge dx_{j_d}$ by the composite τ of the maps (15) for $W = S_t$. Since $dx_{j_1} \wedge \dots \wedge dx_{j_d}$ generate the $\mathcal{O}_{S_t}$-module $\Omega^d_{S_t}$ it follows that $\tau(dx_{j_1} \wedge \dots \wedge dx_{j_d})$ generate the $\mathcal{O}_Z$-module $\tau(\Omega^d_{S_t})$. If $(j_1, \dots, j_d) = (1, \dots, d)$ then we have that $\tau(dx_1 \wedge \dots \wedge dx_d)$ is equal to $X^{e_1 + \dots + e_d}$ times a non zero constant, hence $e_1 + \dots + e_d$ belongs to $J_d(S_t)$. Otherwise, we have $(j_1, \dots, j_d) = (1, \dots, \hat{k}, \dots, d+1)$, for some integer $1 \leq k \leq d$. We write:

$$\tau(dx_{j_1} \wedge \dots \wedge dx_{j_d}) = \psi^- + \psi^+, \text{ with } \psi^\pm := \phi \circ \wedge^d \eta(dX^{\bar{e}^1} \wedge \dots \wedge \widehat{dX^{\bar{e}^k}} \wedge \dots \wedge dX^{\bar{e}^d} \wedge d\zeta_{t,k}^\pm).$$

By definition we have that $\eta(d\zeta_{t,k}^-) = \sum_{\lambda_{m_k} \not\leq \lambda} c_\lambda X^\lambda \otimes \lambda$ and $\eta(d\zeta_{t,k}^+) = \sum_{\lambda_{m_k} \leq \lambda} c_\lambda X^\lambda \otimes \lambda$.

Remark that $\psi^- \in \mathbf{C}[[\sigma^\vee \cap M]]$ is divisible by $X^{\bar{e}^1 + \dots + \bar{e}^d}$ and ψ^+ is of the form $X^\mu \cdot$unit, where $\mu = \lambda_{m_k} + \sum_{i=1,\dots,d}^{i \neq k} \bar{e}^i$. We deduce from this that the inclusion $\subseteq$ holds in Formula (17). The other inclusion follows from the fact that the ideal defined by the right hand side of Formula (17) is generated by $\{\sum_{i=1}^d \bar{e}^i\} \cup \{\lambda_{m_k} + \sum_{i=1,\dots,d}^{i \neq k} \bar{e}^i\}_{m(k) \neq g+1}$. $\square$.

3.3. Comparison with the jacobian ideal of S_t and consequences.

Theorem 3.4. *With the hypothesis and notations of Theorem 3.3, set* $\gamma^0 := \sum_{i=1}^g n_i \bar{e}^{d+i} - \sum_{j=1}^{d+g} \bar{e}^j \in M$. *We denote by* $Jac(S_t)$ *the jacobian ideal of* S_t. *Then the following relation holds:*

$$Jac(S_t)\mathcal{O}_Z = X^{\gamma^0} J_d(S_t)\mathcal{O}_Z. \tag{18}$$

Before proving Theorem 3.4 we describe some consequences and auxiliary results.

Remark 3.5. *The* generating series $P_\Gamma(X) := \sum_{\gamma \in \Gamma} X^\gamma$ of the semigroup Γ *has a rational form, namely,* $P_\Gamma(X) := \prod_{k=1}^g (1 - X^{n_k \bar{e}^{d+k}}) \prod_{k=1}^{d+g} (1 - X^{\bar{e}^k})^{-1}$ *(see* [GP-H]*). We observe the following symmetry property with respect to the vector* γ^0 *defined in Theorem 3.4*

$$P_\Gamma(X) = (-1)^d X^{\gamma_0} P_{-\Gamma}(X).$$

Remark that by Proposition 2.5, the variety Z^Γ *is a complete intersection, hence* $\mathbf{C}[\Gamma]$ *is a* M*-graded Cohen Macaulay Gorenstein domain (cf. with* [Sta] *Chapter 1, Theorem 12.7).*

Corollary 3.6. *The composite of the normalization map of* S_t *with the normalized blow up of* $J_d(S_t)\mathcal{O}_Z$ *is the normalized Nash modification of* S_t.

Proof of Corollary 3.6. Since S_t is a complete intersection the Nash modification of S_t is isomorphic to the blow up of the jacobian ideal in $\mathcal{O}_{S_t}$ (see [No]). Then the normalized Nash modification of S_t is equal to the composition of the normalization map with the normalized blow up of $\mathrm{Jac}(S_t)\mathcal{O}_Z$. This is a general fact concerning the properties of the normalized blow up of an ideal (see [LJ-T], Propositions 3.2 and 3.3). Then the statement follows from Theorem 3.4 since the ideals $\mathrm{Jac}(S_t)\mathcal{O}_Z$ and $J_d(S_t)\mathcal{O}_Z$ are related by invertible monomial ideals they have isomorphic blow ups. ☐

Notation 3.7. *We denote by J_t (resp. $\tilde{J}_t$) the matrix with coefficients in $\mathbf{C}[[\sigma^\vee \cap M]]$ defined from the jacobian matrix of $(H_1, \dots, H_g)$ by:*

$$J_t := \left(\bar{\iota}(\frac{\partial H_i}{\partial U_j}) \right)_{j=1,\dots,d+g}^{i=1,\dots,g}, \quad (\text{resp. } \tilde{J}_t := \left(\bar{\iota}(U_j \frac{\partial H_i}{\partial U_j}) \right)_{j=1,\dots,d+g}^{i=1,\dots,g}),$$

where $\bar{\iota}$ is defined by (14).

Notice that for $t = 0$ the relations (10) correspond by (2) to the element $k(i) := n_i \bar{\lambda}_i$, since $\bar{\iota}(U_{d+i}^{n_i}) = X^{n_i \bar{\lambda}_i}$, for $i = 1, \dots, g$. By Proposition 2.9 all the terms appearing in H_i have weight $\geq n_i \bar{\lambda}_i$ with equality only for those appearing in h_i. We deduce from this and logarithmic differentiation that $X^{n_i \bar{\lambda}_i}$ divides the entries of the i^{th}-row of $\tilde{J}_t$, for $i = 1, \dots, g$.

Definition 3.8. *We denote by R_t the matrix obtained from $\tilde{J}_t$ by factoring out from the entries of the i^{th}-row the term $X^{n_i \bar{\lambda}_i}$, for $i = 1, \dots, g$.*

By Proposition 2.9 the matrix R_t is of the form:

$$R_t = \begin{pmatrix} L_1^{(1)} & \cdots & L_d^{(1)} & L_{d+1}^{(1)} & \epsilon_2 X^{\bar{\lambda}^2 - n_1 \bar{\lambda}^1} & 0 & 0 & \cdots & 0 \\ L_1^{(2)} & \cdots & L_d^{(2)} & L_{d+1}^{(2)} & L_{d+2}^{(2)} & \epsilon_3 X^{\bar{\lambda}^3 - n_2 \bar{\lambda}^2} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ L_1^{(g-1)} & \cdots & L_d^{(g-1)} & L_{d+1}^{(g-1)} & L_{d+2}^{(g-1)} & L_{d+3}^{(g-1)} & \cdots & L_{d+g-1}^{(g-1)} & \epsilon_g X^{\bar{\lambda}^g - n_{g-1} \bar{\lambda}^{g-1}} \\ L_1^{(g)} & \cdots & L_d^{(g)} & L_{d+1}^{(g)} & L_{d+2}^{(g)} & L_{d+3}^{(g)} & \cdots & L_{d+g-1}^{(g)} & L_{d+g}^{(g)} \end{pmatrix}$$

Notice that R_0 is the matrix of relations associated to Z^Γ. If $t \neq 0$ then ϵ_{r+1} is a unit, for $r = 1, \dots, g-1$ and the coefficients of the i^{th}-row have expansions of the form:

$$L_j^{(i)} = l_j^{(i)} + \sum_{\alpha \geq \bar{e}^j, \alpha > n_i \bar{\lambda}^i} c_{j,\alpha}^{(i)} X^{\alpha - n_i \bar{\lambda}^i} \in \mathbf{C}[[\sigma^\vee \cap M]] \text{ for } j = 1, \dots, d+i \tag{19}$$

with $c_{i,\alpha}^{(r)}$ units in $\mathbf{C}[[\sigma^\vee \cap M]]$. More precisely, the terms $c_{i,\alpha}^{(r)} X^{\alpha - n_r \bar{\lambda}^r}$ correspond bijectively with the images in $\mathbf{C}[[\sigma^\vee \cap M]]$ of the terms $U_1^{s_1} \dots U_{d+r}^{s_r}$, of weight $> n_r \bar{\lambda}^r$ appearing in the expansion of H_r, such that $s_i \geq 1$ (in order to have a non zero U_i-derivative).

Lemma 3.9. *If $1 \leq i < m_k$ the monomial $X^{\bar{e}^k}$ divides $L_k^{(i)}$, for $k = 1, \dots, d$ (see Notation 2.6).*

Proof. By definition of m_k we have that $l_k^{(i)} = 0$. By Lemma 2.7 we deduce that the k^{th} coordinate of $n_i \bar{\lambda}^i$ is 0 and if $\alpha \in M_i$ verifies that $\alpha \geq e^k$ and $\alpha \geq n_i \bar{\lambda}^i$ then we have $\alpha - n_i \bar{\lambda}^i \geq e^k$. Then the assertion follows from the expression (19) for $L_k^{(i)}$. ☐

Proof of Theorem 3.4. For any sequence $1 \leq j_1 < \cdots < j_d \leq d+g$ we have that:

$$X^{\bar{e}^1 + \cdots + \bar{e}^{d+g}} \, |J_{j_1,\dots,j_d}| = X^{n_1 \bar{\lambda}^1 + \cdots + n_g \bar{\lambda}^g} X^{\bar{e}^{j_1} + \cdots + \bar{e}^{j_d}} \, |(R_t)_{j_1,\dots,j_d}|. \tag{20}$$

Formula (20) implies that the image of the jacobian ideal of S_t in $\mathcal{O}_Z = \mathbf{C}[[\sigma^\vee \cap M]]$, which is defined by $(|J_{j_1,\dots,j_d}|)_{1 \leq j_1 < \dots < j_d \leq d+g}$, is equal to the ideal generated by:

$$X^{\gamma^0 + \sum_{r=1}^{d} \bar{e}^{j_r}} \, |(R_t)_{j_1,\dots,j_d}|, \text{ for } 1 \leq j_1 < \dots < j_d \leq d+g.$$

Theorem 3.4 follows from the Proposition:

Proposition 3.10.

$$J_d(S_t)\mathcal{O}_Z = (X^{\bar{e}^{j_1}+\dots+\bar{e}^{j_d}} \, |(R_t)_{j_1,\dots,j_d}|)_{1 \leq j_1 < \dots < j_d \leq d+g} \mathcal{O}_Z.$$

Proof. We denote by $\mathcal{J}'_t$ the ideal $(X^{\bar{e}^{j_1}+\dots+\bar{e}^{j_d}} \, |(R_t)_{j_1,\dots,j_d}|)_{1 \leq j_1 < \dots < j_d \leq d+g} \mathbf{C}[[\sigma^\vee \cap M]]$. Notice that we have to prove the assertion only for $t \neq 0$, since for $t = 0$ it is a direct consequence of Proposition 1.5. From now on we suppose that $t \neq 0$. We show first that $\mathcal{J}'_t$ is a monomial ideal contained in $J_d(S)$. We distinguish two essentially different cases for the sequence $j_1, \dots, j_d$.

The case $\bar{e}^{j_1} \wedge \dots \wedge \bar{e}^{j_d} \neq 0$. By Lemma 1.1 the determinant $|(R_t)_{j_1,\dots,j_d}|$ is a unit of $\mathbf{C}[[\sigma^\vee \cap M]]$ with constant term equal to $|(R_0)_{j_1,\dots,j_d}| \neq 0$. We distinguish three possibilities for the sequence $j_1, \dots, j_d$:

a. If $(j_1, \dots, j_d) = (1, \dots, d)$ then we have that $\bar{e}^1 + \dots + \bar{e}^d$ belongs to $J_d(S_t)$.
b. Suppose that the sequence $j_1, \dots, j_d$ is of the form: $j_d = d + k$ and $j_1, \dots, j_{d-1}$ among $1, \dots, d$. Then, we have that $\bar{e}^{d+k} \geq \lambda^k$ hence $\sum_{r=1}^{d} \bar{e}^{j_r} \geq \sum_{r=1}^{d-1} \bar{e}^{j_r} + \lambda_k$. This implies that $\sum_{r=1}^{d} \bar{e}^{j_r} \in J_d(S_t)$ since $\bar{e}^{j_1} \wedge \dots \wedge \bar{e}^{j_d} \neq 0$ if and only if $\bar{e}^{j_1} \wedge \dots \wedge \bar{e}^{j_{d-1}} \wedge \lambda^k \neq 0$ by (8).
c. Otherwise, up to relabelling the vectors $\bar{e}^1, \dots, \bar{e}^d$, we assume in order to simplify the notations that:

$$(j_1, \dots, j_d) = (1, \dots, d-k, d+i_1, \dots, d+i_k), \text{ for } 1 \leq i_1 < \dots < i_k \leq g. \tag{21}$$

The non vanishing of

$$\bar{e}^1 \wedge \dots \wedge \bar{e}^{d-k} \wedge \bar{\lambda}^{i_1} \wedge \dots \wedge \bar{\lambda}^{i_k} \tag{22}$$

implies that at least one of the last k coordinates $(\bar{\lambda}^{i_1}_1, \dots, \bar{\lambda}^{i_1}_d)$ of $\bar{\lambda}^{i_1}$ with respect to the basis $\mathcal{B}$ is non zero. Applying the same argument inductively, we can assume up to relabelling $\bar{e}^1, \dots, \bar{e}^k$, that the $d-r+1, \dots, d$-coordinates of $\bar{\lambda}^{i_r}$ with respect to the basis $\mathcal{B}$ are non zero, for $r = 1, \dots, k$. We deduce the following relation by Lemma 2.8:

$$n_{i_{k-1}} \bar{\lambda}^{i_{k-1}} \geq \bar{e}^{d-k+2} + \dots + \bar{e}^d. \tag{23}$$

By (8) and Lemma 2.8 we have also that:

$$\bar{\lambda}^{i_{k-1}} + \bar{\lambda}^{i_k} \quad \geq \quad n_{i_{k-1}} \bar{\lambda}^{i_{k-1}} + \lambda^{i_k}. \tag{24}$$

We deduce from (23) and (24) that $\sum_{r=1}^{d-k} \bar{e}^r + \sum_{r=1}^{k} \bar{\lambda}^{i_r} \geq \sum_{r=1}^{d-k} \bar{e}^r + \lambda^{i_k} + \sum_{r=d-k+2}^{d} \bar{e}^r \in J_d(S_t)$, since the $(d-k+1)^{th}$-coordinate of λ^{i_k} does not vanish.

The case $\bar{e}^{j_1} \wedge \dots \wedge \bar{e}^{j_d} = 0$. In this case we have that $|(R_0)_{j_1,\dots,j_d}| = 0$ by Lemma 1.1. Without loss of generality we may assume that the sequence $j_1, \dots, j_d$ is of the form (21). We distinguish the following possibilities:

The case $k=1$. Notice that $\bar{e}^1\wedge\cdots\wedge\bar{e}^{d-1}\wedge\bar{\lambda}^{i_1}=0$ implies that $\bar{\lambda}^{i_1}$ belongs to the linear subspace spanned by $\bar{e}^1,\dots,\bar{e}^{d-1}$. This implies that $m_d>k$, for m_d the index defined in Notation 2.6, and that $(R_0)^{m_d,\dots,g}_{1,\dots,d-1,d+1,\dots,d+g}=0$. We consider the expansion of the determinant $|(R_t)_{1,\dots d-1,d+i_1}|$ by its first column: $|(R_t)_{1,\dots,d-1,d+i_1}|=\sum_{r=1}^g \pm L_d^{(r)}\,|(R_t)^r_{1,\dots,d,d+i_1}|$. Notice that if $1\le r\le m_d-1$ then X^{e^d} divides $L_d^{(r)}$ by Lemma 3.9, hence it is sufficient to prove that $X^{\bar{e}^{j_1}+\cdots+\bar{e}^{j_d}}|(R_t)^r_{1,\dots,d,d+i_1}|$ belongs to the monomial ideal $J_d(S_t)$, for $m_d\le r\le g$. If $m_d\le r\le g$ the matrix $(R_t)^r_{1,\dots,d,d+i_1}$ is lower triangular by blocks with three blocks in the diagonal:

$$A_r:=(R_t)^{i_1,\dots,g}_{1,\dots,d,d+i_1,\dots,d+g},\ B_r:=(R_t)^{1,\dots,i_1-1,r,\dots,d+g}_{1,\dots,d+i_1,d+r,\dots,d+g}\text{ and }C_r:=(R_t)^{1,\dots,r}_{1,\dots,d+r}.$$

Notice that $|A_r|=(R_0)^{i_1,\dots,g}_{1,\dots,d,d+i_1,\dots,d+g}+\cdots$, (resp. $|C_r|=(R_0)^{1,\dots,r}_{1,\dots,d+r}+\cdots$), is a unit in $\mathbf{C}[[\sigma^\vee\cap M]]$. The second block is of the form:

$$B_r=\begin{pmatrix}\epsilon_{i_1+1}X^{\bar{\lambda}^{i_1+1}-n_{i_1}\bar{\lambda}^{i_1}} & \cdots & 0\\ \cdots & \cdots & \cdots\\ * & \cdots & \epsilon_r X^{\bar{\lambda}^r-n_{r-1}\bar{\lambda}^{r-1}}\end{pmatrix}$$

Since $\sum_{k=i_1}^{r-1}\bar{\lambda}^{k+1}-n_k\bar{\lambda}^k=\lambda^r-\lambda^{i_1}$ by (8), it follows that $|B_r|=X^{\lambda^r-\lambda^{i_1}}\cdot\text{unit}$, and

$$X^{\sum_{k=1}^d\bar{e}^{j_k}}|(R_t)^r_{1,\dots,d,d+i_1}|=X^{\lambda^r+\sum_{k=1}^{d-1}\bar{e}^k}\cdot\text{unit}.$$

Finally, we notice that $\lambda^r+\sum_{k=1}^{d-1}\bar{e}^k\ge\lambda^{m_d}+\sum_{k=1}^{d-1}\bar{e}_k\in J_d(S_t)$, since $r\ge m_d$.

The case $k>1$ and $\bar{\lambda}^{i_{k-1}}_{d-k+1}\cdots\bar{\lambda}^{i_{k-1}}_d\neq0$. By Lemma 2.8 we have that $n_{i_{k-1}}\bar{\lambda}^{i_{k-1}}\ge\sum_{r=d-k+1}^d\bar{e}^r$. We deduce that $\bar{\lambda}^{i_k}+\bar{\lambda}^{i_{k-1}}\ge\sum_{r=d-k+1}^d\bar{e}^r$ and $\sum_{r=1}^d\bar{e}^{j_r}\ge\sum_{r=1}^d\bar{e}^r\in J_d(S_t)$.

The case $k>1$ and $\bar{\lambda}^{i_{k-1}}_{d-k+1}\cdots\bar{\lambda}^{i_{k-1}}_d=0$. The approach is quite similar to the case $k=1$. Up to permuting the labels of vectors $\bar{e}^{d-k+1},\dots,\bar{e}^d$, we have that:

$$\exists\ 1\le s\le k\ \text{ such that }\ \bar{\lambda}^{i_{k-1}}_{d-k+1}=\cdots=\bar{\lambda}^{i_{k-1}}_{d-k+s}=0\text{ and }\bar{\lambda}^{i_{k-1}}_{d-k+s+1}\cdots\bar{\lambda}^{i_{k-1}}_d\neq0. \tag{25}$$

Since $k\ge2$ we have the following relation by (23) and (24) and Lemma 2.8:

$$\bar{\lambda}^{i_k}+\bar{\lambda}^{i_{k-1}}\ge\lambda^{i_k}+\bar{e}^{d-k+s+1}+\cdots+\bar{e}^d. \tag{26}$$

Since $\bar{\lambda}^1<\cdots<\bar{\lambda}^g$ and the vector (22) vanishes we deduce from (25) that $\bar{\lambda}^j_{d-k+1}=\cdots=\bar{\lambda}^j_{d-k+s}=0$, for $j=1,\dots,i_k$. This implies that:

$$R^{i_k+1,\dots,g}_{1,\dots,d-k,d-k+s+1,\dots,d+g}=0, \tag{27}$$

and that the integer m_{d-k+j}, introduced in Notation 2.6, is $>i_k$, for $j=1,\dots,s$. The expansion of the determinant of the matrix $(R_t)_{1,\dots,d-k,d+i_1,\dots,d+i_k}$ by its first column is of the form $|(R_t)_{1,\dots,d-k,d+i_1,\dots,d+i_k}|=\sum_{r_1=1}^g\pm L^{(r)}_{d-k+1}\,|(R_t)^{r_1}_{1,\dots,d-k+1,d+i_1,\dots,d+i_k}|$. For each $1\le r_1\le i_k$, we expand the determinant of the associated term by the first column obtaining a sum of terms in $1\le r_2\le g$, $r_2\neq r_1$ and then we iterate the expansions of the terms if $1\le r_2\le i_k$, etc. The result of this procedure for the first s-columns is of the form:

$$c|(R_t)_{1,\dots,d-k,d+i_1,\dots,d+i_k}| = \tag{28}$$

$$\sum_{j=1}^{s} \sum_{r_1\le i_k, r_2\le i_k, r_{j-1}\le i_k,}^{r_2\neq r_1,\dots,r_{j-1}\neq r_{j-2},\dots,r_1} \sum_{r_j<m_j} \pm L^{(r_1)}_{d-k+1}\cdots L^{(r_j)}_{d-k+j}\,|(R_t)^{r_1,\dots,r_j}_{1,\dots,d-k+j,d+i_1,\dots,d+i_k}| \tag{29}$$

$$+\sum_{j=1}^{s} \sum_{r_1\le i_k, r_2\le i_k, r_{j-1}\le i_k,}^{r_2\neq r_1,\dots,r_{j-1}\neq r_{j-2},\dots,r_1} \sum_{m_j\le r_j\le g} \pm L^{(r_1)}_{d-k+1}\cdots L^{(r_j)}_{d-k+j}\,|(R_t)^{r_1,\dots,r_j}_{1,\dots,d-k+j,d+i_1,\dots,d+i_k}|. \tag{30}$$

We analyze the terms appearing in this expansion. By formula (27) and Lemma 3.9, if $r_j < m_j$ (resp. if $m_j \le r_j \le g$) the term $L^{(r_1)}_{d-k+1}\cdots L^{(r_j)}_{d-k+j}$ is divisible by

$$X^{\bar e^{d-k+1}+\dots+\bar e^{d-k+j}} \quad (\text{ resp. by } X^{\bar e^{d-k+1}+\dots+\bar e^{d-k+j-1}}). \tag{31}$$

If $r_j < m_j$ the term $L^{(r_1)}_{d-k+1}\cdots L^{(r_j)}_{d-k+j}\,|(R_t)^{r_1,\dots,r_j}_{1,\dots d-k+j,d+i_1,\dots,d+i_k}|$ appearing in (29) is divisible by $X^{\bar e^1+\dots+\bar e^d} \in J_d(S_t)$.

If $m_{d-k+j} \le r_j \le g$ then the matrix $(R_t)^{r_1,\dots,r_j}_{1,\dots d-k+j,d+i_1,\dots,d+i_k}$ is lower triangular by blocks, with three blocks in the diagonal:

$$A_{r_1,\dots,r_j} := (R_t)^{r_1,\dots,r_{j-1},i_k,\dots,g}_{1,\dots,d-k+j,d+i_k,\dots,d+g},\ B_{r_1,\dots,r_j} := (R_t)^{1,\dots,i_k-1,r_j,\dots,g}_{1,\dots,d+i_k,d+r_j+1,\dots,d+g},\ C_{r_1,\dots,r_j} := (R_t)^{1,\dots,r_j}_{1,\dots,d+r_j}.$$

By Lemma 3.9 and Formula (27) the determinant $|A_{r_1,\dots,r_j}|$ is divisible by $X^{\bar e^{d-k+j+1}+\dots+\bar e^{d-k+s}}$, where the integer s is defined in (25). The second block is lower triangular with diagonal equal to $(\epsilon_{i_k+1}X^{\bar\lambda^{i_k+1}-n_{i_k}\bar\lambda^{i_k}},\dots,\epsilon_{r_j}X^{\bar\lambda^{r_j}-n_{r_j-1}\bar\lambda^{r_j-1}})$. By (8) its determinant is of the form: $X^{\lambda^r-\lambda^{i_k}}\cdot$unit. We have that $|C_{r_1,\dots,r_j}| := (R_0)^{1,\dots,r_j}_{1,\dots,d+r_j} + \cdots$ is a unit in $\mathbf{C}[[\sigma^\vee\cap M]]$. It follows from these observations and (31) that the term $L^{(r_1)}_{d-k+1}\cdots L^{(r_j)}_{d-k+j}\,|(R_t)^{r_1,\dots,r_j}_{1,\dots d-k+j,d+i_1,\dots,d+i_k}|$ appearing in (30) is divisible by X^γ where $\gamma = \lambda^{m_{d-k+j}} + \sum_{l=1,\dots,d}^{l\neq d-k+j}\bar e^l \in J_d(S_t)$.

It remains to prove that $J_d(S) \subset \mathcal{J}_t'$. We have proven that $X^{\bar e^1+\dots+\bar e^d} \in \mathcal{J}_t'$. Let us fix a sequence $1 \le i_1 < \dots < i_{d-1} \le d$ and denote t the smallest integer $1 \le t \le g$ such that $\bar e^{i_1}\wedge\dots\wedge\bar e^{i_{d-1}}\wedge\lambda^t \neq 0$. Up to relabelling $\bar e^1,\dots,\bar e^d$ we can suppose that $(i_1,\dots,i_{d-1}) = (1,\dots,d-1)$ and $t = m_d$. It is sufficient to prove that the monomial $X^{\lambda_{m_d}+\sum_{k=1}^{d-1}\bar e^k}$ belongs to $\mathcal{J}_t'$. By the case $k=1$ above, we have shown that $|(\mathcal{J}_t)_{1,\dots,d-1,d+1}|$ is of the form $P+Q$, where P is the monomial $X^{\bar e^1+\dots\bar e^{d-1}+\lambda^t+\bar\lambda^1-\lambda^1}$ times a unit and Q is divisible by $X^{\bar e^1+\dots+\bar e^d}$. Since $\bar\lambda^1=\lambda^1$ it follows that $X^{\bar e^1+\dots+\bar e^{d-1}+\lambda^t} \in \mathcal{J}_t'$. ☐

3.4. **Example: Non constancy of the analytic invariant $\bar\nu_{\mathbf{Jac}(S_t)}(\mathfrak{m}(S_t))$.** We denote by $\mathfrak{m}(S_t)$ the monomial ideal $\bar\iota(U_1,\dots,U_{d+g}) \subset \mathbf{C}[[\sigma^\vee\cap M]]$, where $\bar\iota$ is defined by (14). By theorem 3.4 we have that the image of the jacobian ideal $\mathrm{Jac}(S_t)$ is a monomial ideal of $\mathbf{C}[[\sigma^\vee\cap M]]$. We give an example of a deformation $\mathcal{S}_t$ with fibers S_t such that $\bar\nu_{\mathrm{Jac}(S_0)}(\mathfrak{m}(S_0)) < \bar\nu_{\mathrm{Jac}(S_t)}(\mathfrak{m}(S_t))$, for $t\neq 0$.

Let ζ be a quasi-ordinary branch with characteristic exponents $\lambda^1=(3/2,0)$, $\lambda^2=(7/4,0)$, $\lambda^3=(2,1/2)$. The characteristic integers are $n_1=n_2=n_3=2$. In order to work with integer coordinates we re-scale the first coordinate by the factor 4 and the second by the factor 2. With this re-scaling the associated semigroup is generated by $\bar e^1=(4,0)$, $\bar e^2=(0,2)$, $\bar e^3=(6,0)$, $\bar e^4=(13,0)$ and $\bar e^5=(27,1)$. The semigroup of the normalization of S_t is $\mathbf{Z}^2_{\ge 0}$ (the normalization $\bar S_t$ is smooth). We have that $e^1=(4,0)$, $e^2=(0,2)$, $e^3=(6,0)$, $e^4=(7,0)$ and $e^5=(8,1)$. By

Theorem 3.4, we have that $\gamma^0 = (46, -1)$ and the image of the jacobian ideal of $S_0 = Z^\Gamma$ (resp. of S_t, for $t \neq 0$) in $Z^{\sigma^\vee \cap M}$ is the monomial ideal corresponding to:

$$\bar{e}^1 + \bar{e}^2 + \gamma^0 = (46, 1),\ \bar{e}^1 + \bar{e}^5 + \gamma^0 = (73, 0), \quad (\text{resp. } e^1 + e^2 + \gamma^0 = (46, 1),\ e^1 + e^5 + \gamma^0 = (54, 0)).$$

We consider the normalized Nash modification of S_t, which is equal to the composite of the normalization map and the normalized blow up $Z_{\Sigma_t} \to Z$ of the ideal $J_d(S_t)\mathcal{O}_Z$. The sheaf of ideals $\mathrm{Jac}(S_t)\mathcal{O}_{Z_{\Sigma_t}}$ is locally principal and the associated divisor has two components $D_{t,i}$, for $i = 1, 2$ corresponding to the vectors $n_{t,1} = (1, 0)$ and $n_{t,2} = (1, 8)$ for $t \neq 0$ while $n_{0,2} = (1, 27)$. We deduce from Proposition 1.6 that if $t \neq 0$, $\bar{\nu}_{\mathrm{Jac}(S_t)}(\mathfrak{m}(S_t)) = 4/54 > \bar{\nu}_{\mathrm{Jac}(S_0)}(\mathfrak{m}(S_0)) = 4/73$.

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